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# A SOLUTION OF THE COLLISIONLESS BOLTZMANN EQUATION USING A DIAGRAM TECHNIQUE

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## ABSTRACT

The diagram technique recently developed by the author (Ref. 1) for the solution of Liouville's equation is extended and suitably modified to cover the case of the collisionless Boltzmann equation for a plasma. The usefulness of the method is demonstrated by two problems: First the influence of a plane-polarized electric wave on the electron distribution function of a low temperature plasma, and secondly the propagation of a small initial disturbance for the case of a plasma which is governed by the Vlasov equation (Ref. 2).

## I. INTRODUCTION

In Ref. 1 a solution of the Liouville equation for an $N$ particle system was found, essentially by expanding the associated Green's function into a Sturm-Liouville series. The result obtained may be stated as follows: the distribution function $f$ at time $t$ is uniquely connected with an arbitrarily prescribed initial distribution function $f_0$ at time $t'$ through a scattering operator, thus

$$f(\mathbf{R}, \mathbf{V}, t) = S f_0 [\mathbf{R} - \mathbf{V}(t - t'), \mathbf{V}, t'] \tag{1}$$

The scattering operator in turn is given by the series

$$S = \sum_{n=0}^{\infty} S_n \tag{2}$$

with $S_0 = 1$ and each $S_n$ for $n \geq 1$ turned out to be a sum of contributions consisting of various products of forces and gradients. The structure of these terms can be expressed by diagrams and each contribution may be written down easily according to the rules given in Ref. 1.

It is to be remarked, however, that a complete solution according to this scheme is more or less useless if it is not supplemented by statistical considerations. Of course an exact solution of Liouville's equation is equivalent to an exact solution of the equations of motion of the $N$ particle system, which is, as is well known, a prohibitive venture.

The case to be considered here is the case of the so-called collisionless Boltzmann equation for a plasma. This equation describes a system of charged particles in which only the influence of the long range forces is taken into account. It can be shown that the collisionless Boltzmann equation is obtained from Liouville's equation with only *one* statistical assumption (Ref. 3). This assumption is that the distribution function for the $N$ particles factorizes into a product of distribution functions for each individual particle[1].

$$\begin{aligned} f(\mathbf{R}, \mathbf{V}, t) &= f(\mathbf{r}_1 \mathbf{r}_2 \cdots \mathbf{r}_N, \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_N\, t) \\ &\equiv f(\mathbf{r}_1, \mathbf{v}_1 t)\, f(\mathbf{r}_2, \mathbf{v}_2 t) \cdots f(\mathbf{r}_N, \mathbf{v}_N t) \end{aligned} \tag{3}$$

Considering $f$ as the probability density for finding particle 1 at $\mathbf{r}_1 \mathbf{v}_1$, a particle 2 at $\mathbf{r}_2 \mathbf{v}_2$, and so on, Eq. (3) is an expression for the assumption that the particles are uncorrelated (the joint probability is equal to the product of the individual probabilities). This assumption introduces errors, of course. It is intuitively clear that the assumption of Eq. (3) should break down under any circumstances if two or more particles come close to each other. At low enough densities the encounter of more than two particles is a rare event and the close encounter of two particles finds its expression in the collision integral (Ref. 4). To maintain the assumption of Eq. (3) for *all* values of $\mathbf{r}_j$ and $\mathbf{v}_j$ therefore simply means to neglect the collision integral altogether. This would be a bad approximation if any appreciable forces would be exerted only during close encounters, as is the case in a neutral gas, for instance. But a different situation exists in the case of a plasma. Here there are predominantly the long-range Coulomb forces between the particles so that the error made by neglecting collisions may presumably be within tolerable limits.

In Section II the solution of the initial-value problem for the collisionless Boltzmann equation will be derived. The method of solution will be patterned after the approach given in Ref. 1. However, owing to the non-linear character of the basic equation, the scheme to be developed will be more complex than that given in Ref. 1, but the diagram representation found there can be extended naturally to cover this case. The advantage of the diagram method will be demonstrated in Section III. Once the rules of the game, that is, the connection between the topological structure of a diagram and the mathematical structure of its algebraic counterpart,

[1] The presence of transverse photons does not change this statement. Only the distribution function has to be suitably modified to include the additional degrees of freedom.

are known, it is only a matter of comparatively simple algebra to obtain explicit expressions for the solution of the collisionless Boltzmann equation in many cases – that is, in cases where the number of possible diagrams representing non-vanishing contributions is not forbiddingly high. The real advantage of a diagram expansion is here, as elsewhere (for instance, the Feynman diagrams), to keep track of a large number of possible contributions so that nothing is forgotten and to see immediately whether a certain contribution actually vanishes as the case may be. Here, as elsewhere, a diagram by itself does not have any physical significance other than that of the algebraic expression for which it stands.

## II. DERIVATION

The collisionless Boltzmann equation in its most general form may be written as follows (Ref. 3, 5):

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_r\right) f_j(\mathbf{r},\mathbf{v},t) = \mathbf{A}_j(\mathbf{r},\mathbf{v},t)\cdot\nabla_v f_j(\mathbf{r},\mathbf{v},t) + \int_{-\infty}^{t} dt' \int d^3r'd^3v' \sum_i \mathbf{B}_{ji}(\mathbf{r},\mathbf{v},t;\mathbf{r}',\mathbf{v}',t')\, f_i(\mathbf{r}',\mathbf{v}',t')\cdot\nabla_v f_j(\mathbf{r},\mathbf{v},t) \tag{4}$$

In this equation the meaning of the various terms is as follows: $f_j(\mathbf{r},\mathbf{v},t)$ is the distribution function for particles of kind $j$ (electrons, ions, etc.). The vector $\mathbf{A}_j$ represents an externally applied force acting on the particles of kind $j$. The integral kernel $\mathbf{B}_{ji}$ stands for the interaction of the particles among each other and is essentially given by a complete solution of Maxwell's equations (Ref. 5). The retardation is properly accounted for by the integral over all times $t'$ earlier than $t$. If retardation is neglected $\mathbf{B}_{ji}$ contains a factor $\delta(t-t')$. Equation (4) is now solved with the following "ansatz", which is nothing else than an ordinary perturbation expansion with respect to $\mathbf{A}$ and $\mathbf{B}$.

$$f_j = f_j^{(0)} + f_j^{(1)} + \cdots \tag{5}$$

so that

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_r\right) f_j^{(0)} = 0 \tag{6}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_r\right) f_j^{(1)} = \mathbf{A}_j\cdot\nabla_v f_j^{(0)} + \int_{-\infty}^{t} dt' \int d^3r'd^3v' \sum_i \mathbf{B}_{ji}(\mathbf{r},\mathbf{v},t;\mathbf{r}',\mathbf{v}',t') \times f_i^{(0)}(\mathbf{r}',\mathbf{v}',t')\cdot\nabla_v f_j^{(0)} \tag{6a}$$

and in general

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_r\right) f_j^{(n)} = \mathbf{A}_j \cdot \nabla_v f_j^{(n-1)}$$

$$+ \int_{-\infty}^{t} dt' \int d^3r' d^3v' \sum_i \mathbf{B}_{ji}(\mathbf{r}, \mathbf{v}, t; \mathbf{r}', \mathbf{v}', t') \sum_{m=0}^{n-1} f_i^{(m)}(\mathbf{r}', \mathbf{v}', t') \cdot \nabla_v f_j^{(n-1-m)} \tag{7}$$

Assume that the distribution functions $f_j$ for some initial time $\tau$ are known

$$f_j(\mathbf{r}, \mathbf{v}, \tau) = f_j^*(\mathbf{r}, \mathbf{v}, \tau) \tag{8}$$

the initial electromagnetic fields are also specified

$$\left.\begin{aligned} \mathbf{E} &= E_0(\mathbf{r}, \mathbf{v}) \\ \mathbf{H} &= \mathbf{H}_0(\mathbf{r}, \mathbf{v}) \end{aligned}\right\} \text{ for } t = \tau \tag{9}$$

This, of course, is tantamount to assuming that not only the distribution function at $t = \tau$ is known but that it is also known for earlier times, since

$$-\frac{e_j}{m_j}\left(\mathbf{E}_0 + \frac{1}{c}\mathbf{v} \cdot \mathbf{H}_0\right) = \int_{-\infty}^{\tau} dt' \int d^3r' d^3v' \mathbf{B}_{ji}(\mathbf{r}, \mathbf{v}, t; \mathbf{r}' \mathbf{v}' t') f_i(\mathbf{r}', \mathbf{v}', t') \tag{10}$$

Without retardation $\mathbf{E}_0$ and $\mathbf{H}_0$ are uniquely given by the initial distribution function $f^*$ alone.

The solution of Eq. (6) together with Eq. (8) is given by

$$f_j^{(0)} = f_j^*[\mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau] \qquad \text{for } t \geq \tau \tag{11}$$

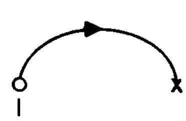
so that the equation for the first-order contribution $f_j^{(1)}$, Eq. (6a), reads

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_r\right) f_j^{(1)} &= \mathbf{A}_j \cdot \nabla_v f_j^*[\mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau] \\ &- \frac{e_j}{m_j}\left(\mathbf{E}_0 + \frac{1}{c}\mathbf{v} \cdot \mathbf{H}_0\right) \cdot \nabla_v f_j^*[\mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau] \\ &+ \int_{\tau}^{t} dt' \int d^3r' d^3v' \sum_i \mathbf{B}_{ji}(\mathbf{r}, \mathbf{v}, t; \mathbf{r}', \mathbf{v}', t') \\ &\times f_i^*[\mathbf{r}' - \mathbf{v}'(t' - \tau), \mathbf{v}', \tau] \cdot \nabla_v f_j^*[\mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau] \end{aligned} \tag{12}$$

From now on the second term on the right hand side of Eq. (12) will be incorporated into the first one without specific change of notation. Eq. (12) is easily solved with the aid of the Green's function introduced in Ref. 1 (see Eq. 3 of Ref. 1) and the first-order contribution is

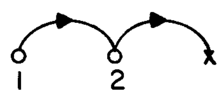
$$f_j^{(1)} = \int_\tau^t dt_1 \Big\{ \mathbf{A}_j \left[ \mathbf{r} - \mathbf{v}(t - t_1), \mathbf{v}, t_1 \right) + \int_\tau^{t_1} dt' \int d^3r' d^3v' \sum_i \boldsymbol{B}_{ji} \left[ \mathbf{r} - \mathbf{v}(t - t_1), \mathbf{v}, t_1; \mathbf{r}', \mathbf{v}', t' \right] \times f_i^* \left[ \mathbf{r}' - \mathbf{v}'(t' - \tau), \mathbf{v}', \tau \right] \Big\} \cdot \left[ \nabla_v - (t_1 - \tau) \nabla_r \right] f_j^* \left[ \mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau \right] \tag{13}$$

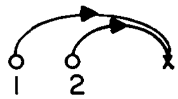
Note that the term involving the external forces **A** is exactly equal to the corresponding term in the expansion of Liouville's equation given in Ref. 1.[2] This, of course, is to be expected since Eq. (4) without the non-linear term is just the one-particle Liouville equation. Now, inserting Eq. (13) back into the equation which expresses $f_j^{(2)}$ by $f_j^{(0)}$ and $f_j^{(1)}$, $f_j^{(2)}$ is easily determined. Continuing along this line, expressions may be found for $f_j^{(3)}$, $f_j^{(4)}$ and so on. In principle, the distribution function is therefore known for all times $t > \tau$, provided it is known together with the initial fields for $t = \tau$. It must be said, however, that owing to the non-linear character of the basic equation, Eq. (4), the higher order terms become rapidly more and more involved so that in practice a general solution is as far away as if Eq. (4) were simply written down and left at that. Fortunately the outlook is not so dim in many cases of interest, namely in cases where some kind of approximations are allowed. But in order to see how exactly any given approximation influences higher order terms $f_j^{(n)}$ it is necessary to study the mathematical structure of a term of arbitrary order. This is conveniently done by means of a diagram technique which allows expression of any contribution to $f_j$ in a concise way. In Ref. 1 a diagram scheme was developed which is applicable to the present problem in its entirety. Provided that the non-linear term of Eq. (4) is missing, the scheme developed in Ref. 1 is completely sufficient and all contributions to any order are given by those diagrams. The first-order contribution, for instance, is given by Eq. (13) if we drop the non-linear ($\boldsymbol{B}$-containing) term. It is represented by the diagram



[2] The properties of the operator $\nabla_v - (t_1 - \tau)\nabla_r$ are explained in Ref. 1. It is noted here that $\nabla_v$ only operates on the second argument $\mathbf{v}$ in $f^* [\mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau]$.

The reader is referred to Ref. 1 for details. The second-order contributions

$\alpha =$ 

$\beta =$ 

(14)

can immediately be written down with the help of the rules given in this reference and it is found that

$$\alpha = \int_\tau^t dt_1 \int_\tau^{t_1} dt_2\, \mathbf{A}\,[\mathbf{r} - \mathbf{v}(t - t_1), \mathbf{v}, t_1] \cdot \mathbf{P}_{12}\, \mathbf{A}\,[\mathbf{r} - \mathbf{v}\,(t - t_2), \mathbf{v}, t_2] \cdot \mathbf{P}_{2\tau} f^* \tag{15}$$

$$\beta = \int_\tau^t dt_1 \int_\tau^{t_1} dt_2\, \mathbf{A}\,[\mathbf{r} - \mathbf{v}(t - t_1), \mathbf{v}, t_1] \cdot \mathbf{P}_{1\tau}\, \mathbf{A}\,[\mathbf{r} - \mathbf{v}\,(t - t_2), \mathbf{v}, t_2] \cdot \mathbf{P}_{2\tau} f^* \tag{16}$$

In these expressions the gradient operations $\mathbf{P}_{\alpha\beta}$ are defined by

$$\mathbf{P}_{\alpha\beta} = \nabla_v - (t_\alpha - t_\beta)\nabla_r \tag{17}$$

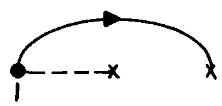
and act on that function of $\mathbf{v}$ and $\mathbf{r}$ on which their representative lines in the corresponding diagram end. In Eq. (15) $\mathbf{P}_{12}$ acts on the succeeding $\mathbf{A}$ vector, whereas $\mathbf{P}_{1\tau} = \nabla_v - (t_1 - \tau)\nabla_r$ acts on the last (external) vertex (that is, on the initial distribution function). From now on a vertex representing an $\mathbf{A}$ vector (external force vector) will be called an $\mathbf{A}$-vertex. The diagrams shown so far contain only $\mathbf{A}$-vertices. A vertex associated with the initial distribution function (the external vertex of Ref. 1) will be called an $f$-vertex. The diagrams shown so far each contain one $f$-vertex. An inspection is now made of the contributions due to the non-linear integral term of Eq. (4). The first-order contribution due to the non-linear integral term is shown in Eq. (13). It is observed that it may be generated from the first contribution (the one represented by an $\mathbf{A}$-vertex) by replacing $\mathbf{A}\,[\mathbf{r} - \mathbf{v}(t - t_1), \mathbf{v}, t_1]$ by $t$

$$\sum_i \int_\tau^{t_1} dt' \int d^3r' d^3v'\, \mathbf{B}_{ji}\,[\mathbf{r} - \mathbf{v}\,(t - t_1), \mathbf{v}, t_1; \mathbf{r}', \mathbf{v}', t']\; f_i^*\,[\mathbf{r}' - \mathbf{v}'(t' - \tau), \mathbf{v}', \tau] \tag{18}$$

It should be noticed, furthermore, that the first set of variables of the integral kernel $B(\mathbf{r}, \mathbf{v}, t; \mathbf{r}', \mathbf{v}', t')$ is treated in exactly the same way as the set of variables of the corresponding $A$-vertex. The vector $\boldsymbol{B}$ is also multiplied by $f^*\,[\mathbf{r}' - \mathbf{v}'\,(t' - \tau), \mathbf{v}', \tau]$. Obviously the zero-order contribution to the distribution function is given by Eq. (11). It may be represented by a simple $f$ vertex

$$\mathrm{x} \equiv f_j^* \; [\mathbf{r} - \mathbf{v}\,(t - \tau), \mathbf{v}, \; \tau] \tag{19}$$

An expression which is mathematically completely equivalent to $\mathbf{A}\,[\mathbf{r} - \mathbf{v}\,(t - t_1), \mathbf{v}, \; t_1]$ in as far as the further steps of calculation are concerned, is obtained by replacing the $\mathbf{r}\mathbf{v}t$ variables of the zero-order term of Eq. (19) by the second set of variables $\mathbf{r}'\mathbf{v}'t'$ of the integral kernel and then integrating over all phase space $d^3r'd^3v'$ and over the time $t'$ from $\tau$ to $t_1$ and finally sum over all distribution functions $i$ as indicated in Eq. (18). A diagram which reproduces these facts is

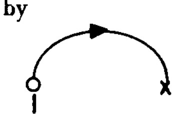


(20)

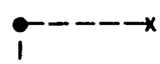
A 'filled dot' is called a $B$-vertex. A $B$-vertex at position $\alpha$ is the representative of the following operator

$$\sum_i \int_\tau^{t_\alpha} dt' \int d^3r'd^3v'\,\mathbf{B}_{ji}\;[\mathbf{r} - \mathbf{v}\,(t - t_\alpha), \mathbf{v}, \; t_\alpha; \mathbf{r}'\!,\mathbf{v}'t'] \cdots \tag{21}$$

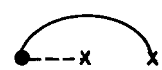
The diagram of Eq. (20) shows a $B$-vertex at position 1. This $B$-vertex is connected with an $f$-vertex by a dotted line. The meaning of this is now clear. The single $f$-vertex which is connected with the $B$-vertex contains the primed variables $\mathbf{r}'\mathbf{v}'t'$ over which the indicated integration of Eq. (21) takes place. In first order, therefore, there are two contributions. The first one is familiar from Ref. 1 and is given by



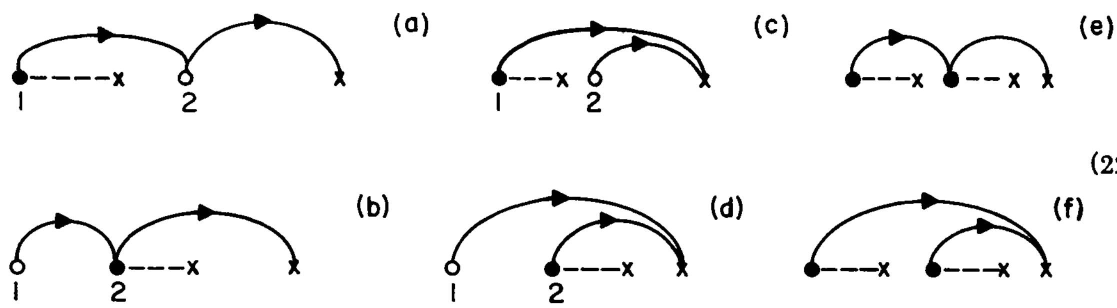
The second one is obtained from the first one by replacing the $A$-vertex by



which is precisely the expression of Eq. (18). It therefore is given by



The second-order contributions may now be investigated. The corresponding diagrams can only contain either two $A$-vertices, two $B$-vertices or one $A$ and one $B$-vertex. The diagrams with two $A$-vertices are shown in Eq. (14). Their contributions are easily obtained by using the rules given in Ref. 1. Replacing either one or both of the $A$-vertices by a $B$-vertex with attached zero-order diagram yields six new possibilities. They are




(22)

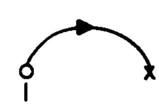
As an example, write the contribution due to the diagram of Eq. (22d). It is

$$\int_\tau^t dt_1 \int_\tau^{t_1} dt_2\, \mathbf{A}_j\, [\mathbf{r} - \mathbf{v}\,(t - t_1), \mathbf{v}, t_1] \cdot \mathbf{P}_{1\tau} \int_\tau^{t_2} dt' \int d^3 r' d^3 v'$$

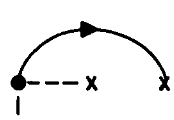
$$\times \sum_i \mathbf{B}_{ji}\, [\mathbf{r} - \mathbf{v}\,(t - t_2), \mathbf{v}, t_2;\, \mathbf{r}', \mathbf{v}', t']\, f_i^*\, [\mathbf{r}' - \mathbf{v}'(t' - \tau), \mathbf{v}', \tau] \tag{23}$$

$$\times \mathbf{P}_{2\tau} f_j^*\, [\mathbf{r} - \mathbf{v}\,(t - \tau), \mathbf{v}, \tau]$$

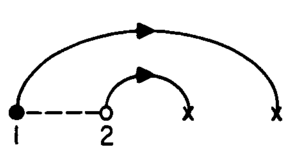
The eight contributions so far considered are not all in the second order. Actually there are two more. A $B$-vertex may have attached to it (by a dotted line) a first-order diagram. Since the $B$-vertex counts as first order, the $B$-vertex with an attached first-order diagram is of second order. Now, there are two first-order diagrams



and

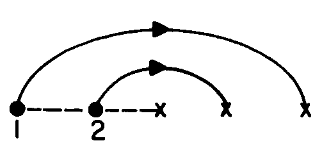


Therefore,



(24a)

and



(24b)

are possible second-order diagrams. Earlier a prescription was given for an $f$-vertex which was connected with a $B$-vertex by a dotted line. It said: multiply the kernel represented by the $B$-vertex with the zero-order diagram in which $\mathbf{r}\,\mathbf{v}$ and $t$ are replaced by the integration variables $\mathbf{r}'\mathbf{v}'$ and $t'$ and then integrate as indicated. Therefore, it is suspected that to obtain the correct expression for the diagram of Eq. (24a), for instance, it is merely necessary to multiply the integral kernel represented by the $B$-vertex with the expression corresponding to the first-order diagram in which only the variables $\mathbf{r}\,\mathbf{v}$ and $t$ are changed into the integration variables $\mathbf{r}'\mathbf{v}'$ and $t'$. This is in fact true. A diagram which is attached to a $B$-vertex by a dotted line is called an internal diagram. It is necessarily of lower order than the complete diagram. Equation (24) shows the two possible cases in which a second-order diagram is constructed by means of $B$-vertices and internal diagrams of the first order. The internal diagram of Eq. (24a) is given by (replacing $\mathbf{r}, \mathbf{v}, t$ by $\mathbf{r}', \mathbf{v}', t'$)

$$\int_{\tau}^{t'} dt_2\, \mathbf{A}_j\left[\mathbf{r}' - \mathbf{v}'(t' - t_2), \mathbf{v}', t_2\right] \cdot \mathbf{P}_{2\tau} f_j^*\left[\mathbf{r}' - \mathbf{v}'(t' - \tau), \mathbf{v}', \tau\right]$$

This is, therefore, the expression with which the kernel (the $B$-vertex) has to be multiplied. Applying these prescriptions to Eq. (24a) it is apparent that it represents

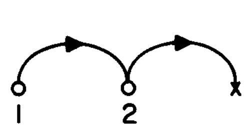
$$\int_{\tau}^{t} dt_1 \sum_i \int_{\tau}^{t_1} dt' \int d^3 r' d^3 v'\, \mathbf{B}_{ji}\left[\mathbf{r} - \mathbf{v}(t - t_1), \mathbf{v}, t_1; \mathbf{r}', \mathbf{v}', t'\right]$$

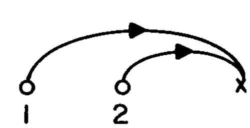
$$\times \int_{\tau}^{t'} dt_2\, \mathbf{A}_i\left[\mathbf{r}' - \mathbf{v}'(t' - t_2), \mathbf{v}', t_2\right] \cdot \mathbf{P}_{2\tau} f_i^*\left[\mathbf{r}' - \mathbf{v}'(t' - \tau), \mathbf{v}', \tau\right] \qquad (25)$$

$$\mathbf{P}_{1\tau} f_j^*\left[\mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau\right]$$

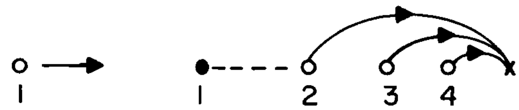
Note here that the expression

may be considered as a replacement for an $A$-vertex so that the rules governing the connection of $A$-vertices by solid lines as outlined in Ref. 1 still apply in their entirety if a simple $A$-vertex is replaced by a more complicated structure (a $B$-vertex with an attached internal diagram). For instance, from the two possible second-order diagrams with only $A$-vertices (Eq. 14)



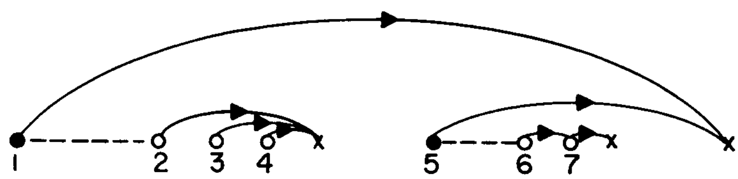



by replacing




and


two possible seventh-order diagrams are obtained, viz:




and


Turning back to the diagram of Eq. (24b), the expression it represents is obtained by noting that the internal diagram it contains is given by Eq. (13) or

$$\int_\tau^t dt_2 \int_\tau^{t_2} dt' \int d^3r'd^3v' \sum_i \mathbf{B}_{ji}\,[\mathbf{r}-\mathbf{v}\,(t-t_2), \mathbf{v},\ t_2;\ \mathbf{r}', \mathbf{v}', t']$$

$$\times\ f_i^*\,[\mathbf{r}'-\mathbf{v}'(t'-\tau), \mathbf{v}', \tau]\cdot \mathbf{P}_{2\tau} f_j^*\,[\mathbf{r}-\mathbf{v}\,(t-\tau), \mathbf{v}, \tau]$$

It is necessary only to replace $\mathbf{r}\mathbf{v}t$ in the above expression by $\mathbf{r}''\ \mathbf{v}''\ t''$, the integration variables of the first $B$-vertex of the diagram Eq. (24b), multiply and integrate thus

$$\int_\tau^{t_1} dt'' \int d^3r''d^3v'' \sum_j \mathbf{B}_{kj}\,[\mathbf{r}-\mathbf{v}\,(t-t_1), \mathbf{v},\ t_1;\ \mathbf{r}'', \mathbf{v}'', t'']$$

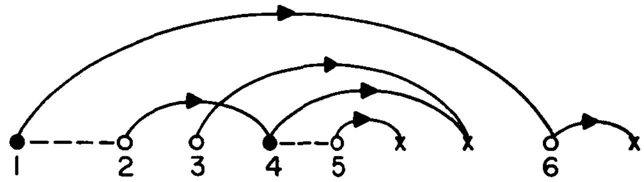
$$\times \int_\tau^{t''} dt_2 \int_\tau^{t_2} dt' \int d^3r'd^3v' \sum_i \mathbf{B}_{ji}\,[\mathbf{r}''-\mathbf{v}''(t''-t_2), \mathbf{v}'', t_2;\ \mathbf{r}', \mathbf{v}', t']$$

$$\times\ f_i^*\,[\mathbf{r}'-\mathbf{v}'(t'-\tau), \mathbf{v}', \tau]\cdot \mathbf{P}_{2\tau} f_j^*\,[\mathbf{r}''-\mathbf{v}''(t''-\tau), \mathbf{v}'', \tau]$$

to obtain the expression which replaces the simple $A$-vertex of the diagram

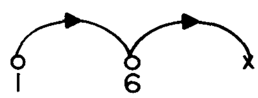

thereby converting it into the diagram of Eq. (24b).

For the sake of completeness,the rules for the association of given diagram with its corresponding contribution to the distribution function may be illustrated with the following example:



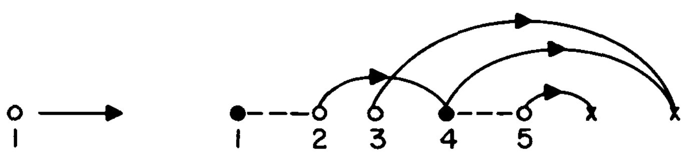

(26)

This is a sixth-order diagram which contains two internal diagrams. It is generated from the simple second-order diagram



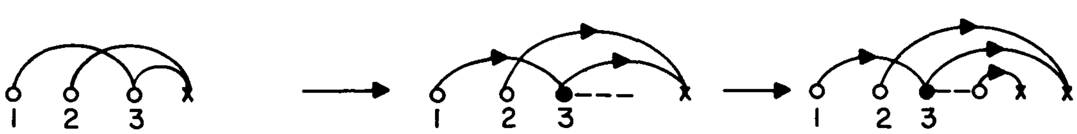

(27)

by replacing the first $A$-vertex with a $B$-vertex with an attached internal diagram of the fourth-order, thus



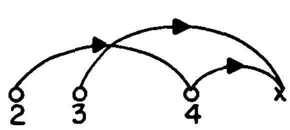

(28)

The fourth-order diagram in turn is obtained from an third-order diagram by replacing the third $A$-vertex with a $B$-vertex with attached first-order diagram in the following way:




(29)

In order to write down the contribution to the distribution function $f$ represented by the diagram of Eq. (26) it is first necessary to work out the internal diagram of Eq. (28). The diagram



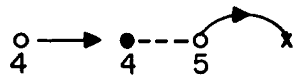

(30)

gives (applying the rules of Ref. 1)

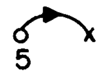
$$\int_{\tau}^{t} dt_2 \int_{\tau}^{t_2} dt_3 \int_{\tau}^{t_3} dt_4 \, \mathbf{A}\,[\mathbf{r} - \mathbf{v}\,(t - t_2), \mathbf{v}, t_2] \cdot \mathbf{P}_{24}\, \mathbf{A}\,[\mathbf{r} - \mathbf{v}\,(t - t_3), \mathbf{v}, t_3]$$

$$\cdot\, \mathbf{P}_{3\tau}\, \mathbf{A}\,[\mathbf{r} - \mathbf{v}\,(t - t_4), \mathbf{v}, t_4] \cdot \mathbf{P}_{4\,\tau}\, f_j^{*}\,[\mathbf{r} - \mathbf{v}\,(t - \tau), \mathbf{v}, \tau]$$

Now the $A$-vertex number 4 must be replaced by a $B$-vertex with attached first-order diagram

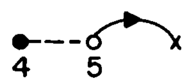



To do this the first-order diagram is written




$$\equiv \int_\tau^t dt_5 \, \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - t_5), \mathbf{v}, t_5\right] \cdot \mathbf{P}_{5\tau} f_j^* \left[\mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau\right]$$

Therefore




is given by (according to Eq. 21)

$$\int_\tau^{t_4} dt' \int d^3r' d^3v' \sum_i \mathbf{B}_{ji}\left[\mathbf{r} - \mathbf{v}(t - t_4), \mathbf{v}, t_4; \mathbf{r}', \mathbf{v}', t'\right]$$

$$\times \int_\tau^{t'} dt_5 \, \mathbf{A}\left[\mathbf{r}' - \mathbf{v}'(t' - t_5), \mathbf{v}', t_5\right] \cdot \mathbf{P}'_{5\tau} f_i^* \left[\mathbf{r}' - \mathbf{v}'(t' - \tau), \mathbf{v}', \tau\right]$$

This, then, is the expression which replaces the $A$-vertex number 4 in Eq. (30). In other words, the complete internal diagram of Eq. (29) is given by

$$\alpha_j(\mathbf{r}, \mathbf{v}, t) = \int_\tau^t dt_2 \int_\tau^{t_2} dt_3 \int_\tau^{t_3} dt_4 \, \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - t_2), \mathbf{v}, t_2\right] \cdot \mathbf{P}_{24}$$

$$\times \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - t_3), \mathbf{v}, t_3\right] \cdot \mathbf{P}_{3\tau} \int_\tau^{t_4} dt' \int d^3r' d^3v' \sum_i \mathbf{B}_{ji}$$

$$\left[\mathbf{r} - \mathbf{v}(t - t_4), \mathbf{v}, t_4; \mathbf{r}', \mathbf{v}', t'\right] \int_\tau^{t'} dt_5 \, \mathbf{A}\left[\mathbf{r}' - \mathbf{v}'(t' - t_5), \mathbf{v}', t_5\right] \cdot \mathbf{P}'_{5\tau} f_i^*$$

$$\left[\mathbf{r}' - \mathbf{v}'(t' - \tau), \mathbf{v}', \tau\right] \cdot \mathbf{P}_{4\tau} f_j^* \left[\mathbf{r} - \mathbf{v}(t - \tau), \mathbf{v}, \tau\right]$$

Again applying the rule of Eq. (21), this time to Eq. (28), yields the expression

$$\int_\tau^{t_1} dt'' \int d^3r'' d^3v'' \sum_j \mathbf{B}_{kj}\left[\mathbf{r} - \mathbf{v}(t - t_1), \mathbf{v}, t_1; \mathbf{r}'', \mathbf{v}'', t''\right] \alpha_j(\mathbf{r}'', \mathbf{v}'', t'')$$

which replaces the first $A$-vertex of the diagram of Eq. (27). Since the diagram of Eq. (27) is given by

$$\int_{\tau}^{t} dt_1 \int_{\tau}^{t_1} dt_6\, \mathbf{A}\,[\mathbf{r}-\mathbf{v}\,(t-t_1), \mathbf{v}, t_1] \cdot \mathbf{P}_{16} \cdot \mathbf{A}\,[\mathbf{r}-\mathbf{v}\,(t-t_6), \mathbf{v}, t_6] \cdot \mathbf{P}_{6\tau}\, f_k^*\,[\mathbf{r}-\mathbf{v}\,(t-\tau), \mathbf{v}, \tau]$$

it is found that for the diagram of Eq. (26)

$$\begin{aligned}\text{the diagram of Eq. (26)} \equiv \int_{\tau}^{t} dt_1 \int_{\tau}^{t_1} dt_6 \int_{\tau}^{t_1} dt'' \int d^3 r'' d^3 v'' \sum_j \alpha_j\,(\mathbf{r}'', \mathbf{v}'', t'') \\ \times \mathbf{B}_{kj}\,[\mathbf{r}-\mathbf{v}\,(t-t_1), \mathbf{v}, t_1; \mathbf{r}'', \mathbf{v}'', t''] \cdot \mathbf{P}_{16} \\ \times \mathbf{A}\,[\mathbf{r}-\mathbf{v}\,(t-t_6), \mathbf{v}, t_6] \cdot \mathbf{P}_{6\tau}\, f_k^*\,[\mathbf{r}-\mathbf{v}\,(t-\tau), \mathbf{v}, \tau]\end{aligned} \tag{31}$$

Although now it is possible to form a concise idea as to what contributions to expect for any given order $n$, a formula will be given here for the number $N_n$ of diagrams of order $n$.[3] Let $n$ be the order for which it is desired to know the number of possible diagrams $N_n$. Then write

$$n = \alpha_0 + 2\alpha_1 + 3\alpha_2 + \cdots n\alpha_{n-1} \tag{32}$$

and determine all possible ways by which Eq. (32) can be satisfied with positive integers $\alpha_1$ $\alpha_2 \cdots \alpha_{n-1}$. For instance for $n = 4$ we would have the five possible solutions

$$\begin{array}{lll} \alpha_0 = 4 & \alpha_1 = \alpha_2 = \alpha_3 = 0 & \\ \alpha_0 = 2 & \alpha_1 = 1 & \alpha_2 = \alpha_3 = 0 \\ \alpha_0 = 1 & \alpha_1 = 0 & \alpha_2 = 1 \quad \alpha_3 = 0 \\ \alpha_0 = 0 & \alpha_1 = 2 & \alpha_2 = \alpha_3 = 0 \\ \alpha_0 = 0 & \alpha_1 = \alpha_2 = 0 & \alpha_3 = 1 \end{array}$$

[3] The author is grateful to H. Wahlquist for the derivation of the formula in Eq. (33).

If there are $M$ solutions in the general case. there are $M$ sets of positive integers $\alpha_i^{(\epsilon)}$ (including the zero) $1 \leq \epsilon \leq M$. The number $N_n$ is then given by the expression

$$N_n = \sum_{\epsilon=1}^{M} \frac{\left[\left(\sum_{i=0}^{n-1} \alpha_i^{(\epsilon)}\right)!\right]^2}{\prod_{i=0}^{n-1} (\alpha_i^{(\epsilon)})!} \, 2^{\alpha_0^{(\epsilon)}} \prod_{\gamma=0}^{n-1} (N_\gamma)^{\alpha_\gamma^{(\epsilon)}} \tag{33}$$

The sum goes over all possible solutions of Eq. (32). To give an idea of how rapidly $N_n$ increases, $N_n$ is listed for the first few orders. For large $n$, $N_n$ goes approximately as $2^n n!$

| $n$ | $N_n$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 10 |
| 3 | 74 |
| 4 | 690 |

# III. APPLICATIONS

Although the number of diagrams, and therefore the number of contributions, increases tremendously with increasing $n$, nevertheless there are many cases which can be handled advantageously by the diagram method. Here are two examples confined to the Vlasov equation (Ref. 2). Its linearized version has been treated by several authors (Ref. 6, 7, 8). The Vlasov equation is applicable to a moderately low-density, fully ionized, electron-ion plasma in which the ions have negligible velocities (they form an immovable uniform background of positive charges). Only the electrons are considered to move at liberty, but again they are slow enough so that all effects of retardation for the electromagnetic fields are negligible ($v/c \to 0$). Equation (1) of Section II becomes Vlasov's equation if the index of the distribution function is dropped (it is necessary only to be concerned about the electron distribution) and if

$$\mathbf{B}\,(\mathbf{r},\, \mathbf{v},\, t\,;\mathbf{r}',\mathbf{v}',t') \equiv -\frac{e^2 N}{m}\,\delta\,(t-t')\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{34}$$

Here $N$ is the number density of electrons.

The first problem considered here is the following: At $t = \tau = 0$ a plane-polarized light wave is switched on. Initially the electron distribution function $f\,(\mathbf{r},\, \mathbf{v},\, t)$ was Maxwellian $f^* = f^*(\mathbf{v}) \sim e^{-\alpha v^2}$.

How does the light wave disturb the plasma? The light wave may be described by

$$\mathbf{E}_{\text{ext}} = \mathbf{a} \cos\,(\mathbf{k}\cdot\mathbf{r} - \omega t) \tag{35}$$

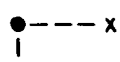
with

$$\mathbf{a}\cdot\mathbf{k} = 0 \tag{36}$$

Vlasov's equation reads then (for $t > 0$)

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_r\right) f = &-\frac{e}{m}\,\mathbf{a}\cdot\nabla_v f\,\cos\,(\mathbf{k}\cdot\mathbf{r} - \omega t)\\ &-\frac{e^2 N}{m}\int d^3r'\,d^3v'\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,f\,(\mathbf{r}',\mathbf{v}',t)\cdot\nabla_v f\end{aligned} \tag{37}$$

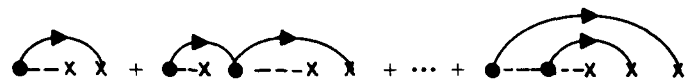
$f(\mathbf{r}, \mathbf{v}, t)$ for $t > 0$ is given by the sum of all possible diagrams according to Section II. It will be immediately noticed that a great simplification arises from the fact that the initial $f$ is solely a function of velocity. Because of this, all diagrams containing a zero-order internal diagram vanish and



$$\equiv \int d^3 r' d^3 v' \frac{\mathbf{r} - \mathbf{v}(t - t_1) - \mathbf{r}'}{|\mathbf{r} - \mathbf{v}(t - t_1) - \mathbf{r}'|^3} f^*(\mathbf{v}') \equiv 0 \tag{38}$$

Without the external force of Eq. (35) the following may be written:

$$f(\mathbf{r}, \mathbf{v}, t) = X +$$



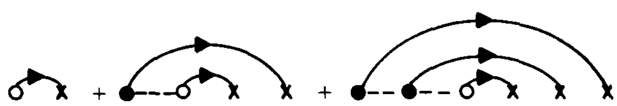
(39)

and it is evident that all diagrams vanish and the general solution is

$$f(\mathbf{r}, \mathbf{v}, t) = f^*(\mathbf{v}) \tag{40}$$

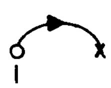
independent of time, provided it was only a function of velocity initially. Now for consideration of the effect of the external force switched on at $t = 0$. If the light wave is sufficiently small, all higher-order terms but the first may be neglected. In other words, only one $A$-vertex representing the light wave equation of Eq. (35) is allowed in any diagram. The only diagrams which do not vanish offhand to this order (because of Eq. (38)) are

$$f(\mathbf{r}, \mathbf{v}, t) = X +$$



(41)

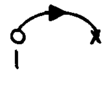
The actual evaluation is easy, and



$$= \int_0^t dt_1 \, \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - t_1), \mathbf{v}, t_1\right] \cdot \boldsymbol{P}_{1\tau} f^*(\mathbf{v})$$

$$\equiv -\frac{e}{m} \int_0^t dt_1 \cos\left\{\boldsymbol{k} \cdot \left[\boldsymbol{r} - \mathbf{v}(t - t_1)\right] - \omega t_1\right\} \boldsymbol{a} \cdot \nabla_v f^*(\mathbf{v}) \tag{42}$$

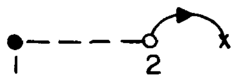
or



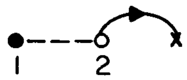
$$= -\frac{e}{m} \boldsymbol{a} \cdot \nabla_v f^*(\mathbf{v}) \, F(\boldsymbol{k} \cdot \mathbf{v}, \boldsymbol{r}, t) \tag{43}$$

$$F(\mathbf{k}\cdot\mathbf{v},\mathbf{r},t) = \frac{\sin(\mathbf{k}\cdot\mathbf{r}-\omega t) - \sin\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}{\mathbf{k}\cdot\mathbf{v}-\omega} \tag{44}$$

In order to evaluate the remaining terms of the series of Eq. (41) the always-occurring combination is determined:



According to the rules of Section II

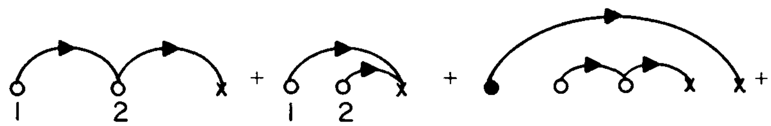


$$\sim \int d^3r'd^3v' \frac{\mathbf{r}-\mathbf{v}(t-t_1)-\mathbf{r}'}{|\mathbf{r}-\mathbf{v}(t-t_1)-\mathbf{r}'|^3} \times F(\mathbf{k}\cdot\mathbf{v}', \mathbf{r}', t_1)\, \mathbf{a}\cdot\nabla_{v'} f^*(\mathbf{v}') \tag{45}$$

where $F$ is that part of the expression of Eq. (43) which depends explicitly on the scalar product $\mathbf{k}\cdot\mathbf{v}$. But by partial integration the integral over $\mathbf{v}'$ can be converted into

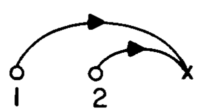
$$\int d^3v' F(\mathbf{k}\cdot\mathbf{v}', \mathbf{r}', t_1)\, \mathbf{a}\cdot\nabla_v f^*(\mathbf{v}) = -\mathbf{a}\cdot\mathbf{k}\int d^3v' f^*(\mathbf{v}') \left(\frac{dF(u, \mathbf{r}', t_1)}{du}\right)_{u=\mathbf{k}\cdot\mathbf{v}'} \equiv 0 \tag{46}$$

which gives zero by virtue of Eq. (36). So it is apparent that to the first order only Eq. (42) contributes to Eq. (41). In the second order there would be contributions from the following series:



$$+ \cdots \tag{47}$$

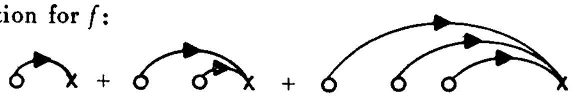
The first diagram of Eq. (47) vanishes again because of Eq. (36) and the second yields:



$$\equiv \frac{1}{2}\left(-\frac{e}{m}\right)^2 \{F(\mathbf{k}\cdot\mathbf{v},\mathbf{r},t)\}^2 (\mathbf{a}\cdot\nabla_v)^2 f^*(\mathbf{v}) \tag{48}$$

Considering Eq. (48) as a possible candidate for an internal diagram it is seen that by the same reasoning as before, no contribution arises. This is easily extended to all higher orders and there appears as solution for $f$:

$$f(\mathbf{r},\mathbf{v},t) = f^*(\mathbf{v}) +$$



$$+ \cdots \tag{49}$$

all other diagrams vanish. It is not difficult to sum these diagrams up. In fact the general $n$th-order term is:

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - t_1), t_1\right] \cdot \nabla_v$$
$$\times \, \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - t_2\right] \cdot \nabla_v \cdots \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - t_n), t_n\right] \cdot \nabla_v f^*(\mathbf{v})$$

which is simply:

$$\frac{1}{n!}\left\{\int_0^t d\tau \, \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - \tau), \tau\right] \cdot \nabla_v\right\}^n f^*(\mathbf{v})$$

The series of Eq. (49) therefore yields

$$\begin{aligned} f(\mathbf{r}, \mathbf{v}, t) &= \exp\left\{\int_0^t d\tau \, \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - \tau), \tau\right] \cdot \nabla_v\right\} f^*(\mathbf{v}) \\ &= f^*\left(\mathbf{v} + \int_0^t d\tau \, \mathbf{A}\left[\mathbf{r} - \mathbf{v}(t - \tau), \tau\right]\right) \end{aligned} \tag{50}$$

Inserting the expression of Eq. (42) into Eq. (50) the desired result is finally obtained:

$$f(\mathbf{r}, \mathbf{v}, t) = f^*\left(\mathbf{v} - \frac{e}{m}\,\mathbf{a}\,\frac{\sin(\mathbf{k}\cdot\mathbf{r} - \omega t) - \sin \mathbf{k}\cdot(\mathbf{r} - \mathbf{v}t)}{\mathbf{k}\cdot\mathbf{v} - \omega}\right) \tag{51}$$

It is easily verified that Eq. (51) is, in fact, a solution of Eq. (37).

The second case to be considered here is treated several times in the literature (Ref. 6, 7, 8). It consists of the following problem: Suppose that initially the distribution function is split into two parts:

$$f^*(\mathbf{r}, \mathbf{v}, \tau) = f_0(\mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}) \tag{52}$$

where the space-dependent part, $f_1$, is considered small compared to the uniform background which is assumed to be Maxwellian,

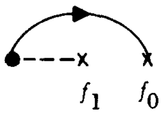
$$f_0(\mathbf{v}) = \left(\frac{\alpha}{\pi}\right)^{\frac{3}{2}} e^{-\alpha \mathbf{v}^2} \qquad \alpha = \frac{m}{2kT} \tag{53}$$

The question is, how does the initial disturbance $f_1$ propagate in time? Since there are no external forces, the solution is entirely given by diagrams consisting of $B$-vertices. It is assumed, furthermore, that all higher order terms of $f_1$ may be neglected. This is equivalent to using the linearized version of the Vlasov equation. Consider now the basic vertex $\bullet\text{---x}$ . From Eq. (38) and Eq. (52) it is seen that

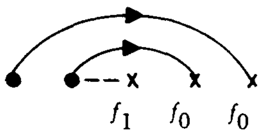
$$\underset{f^*}{\bullet\text{----x}} \equiv \underset{f_0+f_1}{\bullet\text{----x}} = \underset{f_1}{\bullet\text{----x}} \tag{54}$$

so that only $f_1$ survives as a zero-order internal diagram. A complete description of the time development of the distribution function is obtained through the following series:

$$f = f_0 + f_1(\mathbf{r} - \mathbf{v}t, \mathbf{v}) + \tag{55}$$




$+$

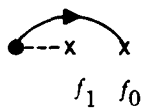



$+ \cdots$

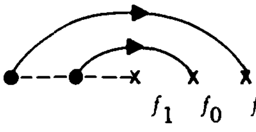
all other diagrams either vanish or give a higher order contribution with respect to $f_1$. For $f_1$ take

$$f_1 = \epsilon\, \delta(\mathbf{v})\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{56}$$

$\epsilon$ measures the strength of the anisotropy. The expression of Eq. (56) really is only a Fourier component of the arbitrary density fluctuation $\zeta(\mathbf{r}) = (2\pi)^{-3} \int \zeta(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3k$, but since the series of Eq. (55) is linear in $f_1$, it may first be summed and then integrated over $\mathbf{k}$ to obtain the result for an arbitrary $\zeta(\mathbf{r})$. The particles represented by $f_1$ are also considered to be at rest initially. After some calculation it is found that:

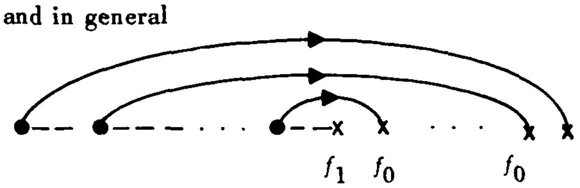



$$\equiv 4\pi i\, \epsilon\, \frac{e^2 N}{m} \int_0^t dt_1\, e^{i\mathbf{k}\cdot[\mathbf{r}-\mathbf{v}(t-t_1)]}\, \frac{\mathbf{k}\cdot\nabla_v f_0(\mathbf{v})}{k^2} \tag{57}$$




$$\equiv (4\pi i)^2\, \epsilon i \left(\frac{e^2 N}{m}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, e^{i\mathbf{k}\cdot[\mathbf{r}-\mathbf{v}(t-t_1)]} \times (t_1 - t_2)\, g\,[(t_1 - t_2)\mathbf{k}]\, \frac{\mathbf{k}\cdot\nabla_v f_0(\mathbf{v})}{k^2} \tag{58}$$

and in general




$$\equiv -i\epsilon\left(-\frac{4\pi e^2 N}{m}\right)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n$$

$$\times\, e^{ik\cdot[r-v(t-t_1)]}\,(t_1-t_2)\,g\,[(t_1-t_2)k]\,(t_2-t_3)\,g\,[(t_2-t_3)k]\cdots \tag{59}$$

$$(t_{n-1}-t_n)\,g\,[(t_{n-1}-t_n)k]\,\frac{k\cdot\nabla_v f_0(\mathbf{v})}{k^2}$$

here $g(k)$ is defined as the Fourier transform of the background distribution function

$$g(tk) = \int d^3v\, f_0(\mathbf{v})e^{-itk\cdot v} = e^{-\frac{k^2}{4\alpha}t^2} \tag{60}$$

from Eq. (53). The general expression of Eq. (59) can be simplified considerably by noting that the time integrals are nothing else than a number of convolution integrals "nested" into each other. Defining the Laplace transform of $e^{-itk\cdot v}$ and $g(tk)$ by

$$L\,(e^{-itk\cdot v}) = \int_0^\infty e^{-st}\,e^{-itk\cdot v}\,dt = (s+ik\cdot \mathbf{v})^{-1} = \alpha(s) \tag{61}$$

$$L\,[g(tk)] = \int_0^\infty e^{-st}\,g(tk)\,dt = \beta(s) = \frac{\sqrt{\pi\alpha}}{k}\,e^{\frac{\alpha s^2}{k^2}}\left\{1-\phi\left(\frac{\sqrt{\alpha}\,s}{k}\right)\right\} \tag{62}$$

we see that the Laplace transform of the general term of Eq. (59) is

$$L\,[\text{Eq. (59)}] = -i\epsilon\left(-\frac{4\pi e^2N}{m}\right)^n e^{ik\cdot r}\times\frac{k\cdot\nabla_v f_0(\mathbf{v})}{k^2}\,\frac{\alpha(s)}{s}\left(\frac{2\alpha}{k^2}\right)^n[1-s\beta(s)]^{n-1} \tag{63}$$

It is therefore easy to sum the Laplace transform of the series of Eq. (55) and the result is:

$$L(f-f_0-f_1) = i\epsilon\,\frac{4\pi e^2N}{m}\,e^{ik\cdot r}\,\frac{k\cdot\nabla_v f_0(\mathbf{v})}{k^2}\,\frac{\alpha(s)}{s}\,[1+(k\lambda_D)^{-2}(1-s\beta(s))]^{-1} \tag{64}$$

where the Debye-Hückel length was introduced

$$\lambda_D = \left(\frac{kT}{4\pi e^2 N}\right)^{\frac{1}{2}} \tag{65}$$

The expression of Eq. (64) can easily be shown to be identical with the result obtained by Landau (Ref. 6) and Berz (Ref. 9).

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